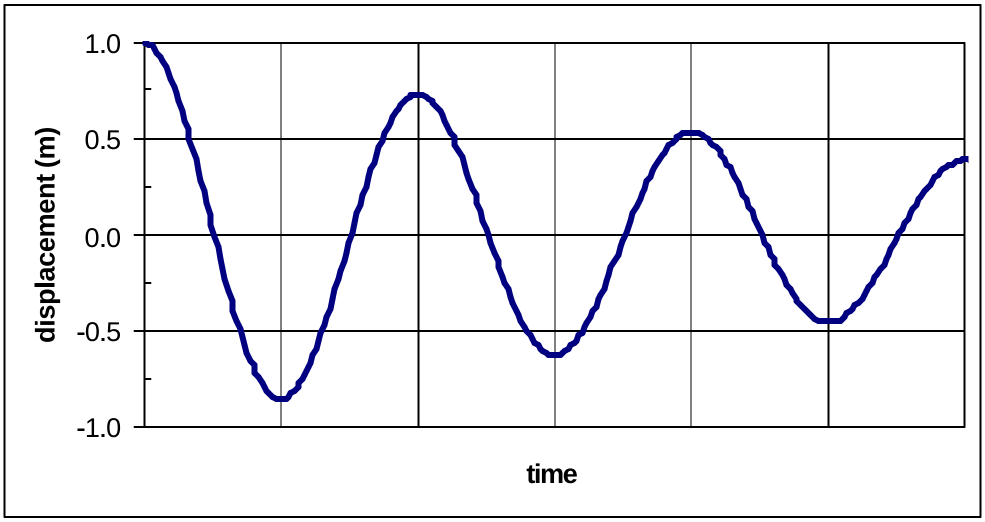
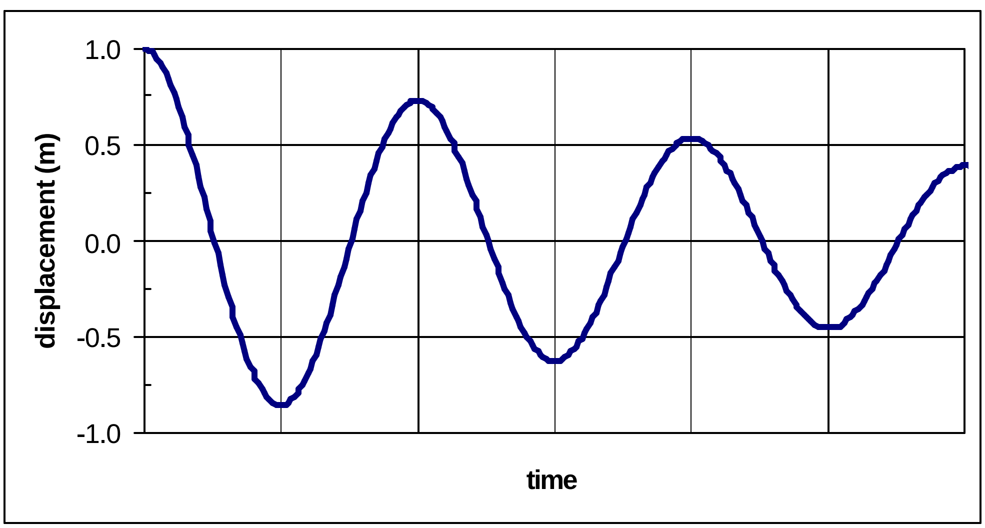
Homework: Damped oscillations: Motion graphs Name ______________________

1. An underdamped oscillator moves such that its position is expressed as a function of time as: $x(t) = A\, e^{-\gamma t} \cos(w_d t + f_o)$. The x *vs.* t graphs in parts a and b below represent the motion of such an oscillator released from rest at $t = 0$.

   a. On the graph below, sketch a qualitatively correct x. *vs*. t graph (drawn to the same scale as the original graph) for an oscillator having the <u>same (damped) frequency</u> as the original oscillator but a <u>larger damping constant</u> *g*. Explain how you decided to draw the new graph.



   b. On the graph below, sketch a qualitatively correct x. *vs*. t graph (drawn to the same scale as the original graph) for an oscillator having the <u>same damping constant</u> as the original oscillator but a <u>larger (damped) frequency</u>. Explain how you decided to draw the new graph.



(Problem 1 continued on next page)

1. [continued]

   c. Consider two underdamped oscillators having the same (damped) frequency but different damping constants (*e.g.,* the two oscillators represented in the x *vs.* t graphs from part a on the preceding page).

   If the damping were removed from both of these oscillators, would they have the <u>same (natural) frequency</u>? If so, explain why. If not, specify which oscillator (the one with the greater or lesser damping) that would have the larger natural frequency, and explain your reasoning.

2. An underdamped oscillator is released from rest at $t = 0$. In this problem we use the function $x(t) = A\,e^{-\gamma t}\cos(w_d t + f_o)$ to represent the position of the oscillator as a function of time.

   a. At $t = 0$, would the slope of the graph of each function below (plotted as a function of time) be *positive, negative,* or *zero?* Explain your reasoning for each case.

      i. the entire function $x(t) = A\,e^{-\gamma t}\cos(w_d t + f_o)$

      ii. just the exponential part of the function, $e^{-\gamma t}$

      iii. just the cosine part of the function, $\cos(w_d t + f_o)$

   b. On the basis of your result in part a.iii, explain why the value of the initial phase angle f_o <u>cannot</u> be equal to zero, and determine whether f_o is (slightly) *positive* or *negative* in value. Explain.

   c. Verify your answer in part b quantitatively: Using the above form of $x(t)$, determine the condition that must be satisfied at those instants t when the oscillator attains maximum displacement (*i.e.,* when $v(t) = 0$), and find an expression for f_o in terms of w_d and g. Show all work. (*Note:* Make sure your answers in parts b and c are consistent!)

      Although f_o is not exactly equal to zero for an oscillator starting from rest, under what limiting conditions would f_o tend toward zero? Explain your reasoning.

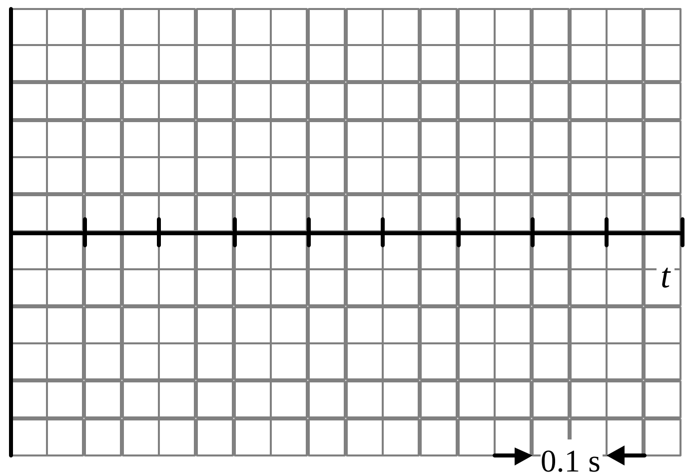
   d. Your results in parts b and c imply that the instants when the cosine function has value +1 or –1 are *not* the same instants when the oscillator achieves maximum displacement. Nevertheless, extend your work in part c to show that the damped oscillator must attain maximum displacement at instants that are $T_d/2$ (or, p/w_d) apart.

3. An underdamped oscillator moves such that its position is expressed as a function of time as: $x(t) = A\,e^{-\gamma t}\cos(w_d t)$. Such an oscillator crosses $x = 0$ whenever the cosine term equal zero, *i.e.,* at times $t = \pi/2w_d$, $3\pi/2w_d$, *etc.*

   Using the form of $x(t)$ given above, show that the first time at which the damped oscillator attains maximum speed occurs <u>before</u> $t = \pi/2w_d$. Carefully show all work and explain your reasoning.

4. Consider the underdamped oscillator from section II of the tutorial. Recall that the oscillator is released from rest at $t = 0$ and that the period of the oscillator is 0.4 sec.

   a. On the set of axes at right, draw two graphs: (i) potential energy vs. time and (ii) kinetic energy vs. time. (Draw the graphs one on top of the other. Use a different color of ink or pencil for each graph.)



   b. At each of the instants $t = 0.1$ s, $t = 0.2$ s, $t = 0.3$ s, and so on, is each of the following quantities instantaneously *increasing*, *decreasing*, or *constant* at that instant? Explain how you can tell from your graphs.

   - potential energy
   - kinetic energy
   - total energy

   c. Are your results in parts a and b above consistent with the fact that, for an underdamped oscillator, the instantaneous rate of energy loss is given by $dE/dt = -c|\dot{x}|^2$? If not, resolve the inconsistencies.

   d. Some textbooks make the claim that, at any instant t, the total energy $E(t)$ of an underdamped oscillator is directly proportional to an exponentially decreasing factor:

$$\text{“}E(t) \propto \exp(-Kt)\text{”}$$

   On the basis of your results in parts a – c above, would you agree *completely* with this claim?

   If so: State so explicitly and explain why. Express the constant K that appears in the above statement in terms of the damping constant g.

   If not: Explain how you would modify or re-interpret the above statement so that it is fully consistent with your results from parts a – c. Express the constant K that appears in the above statement in terms of the damping constant g.

