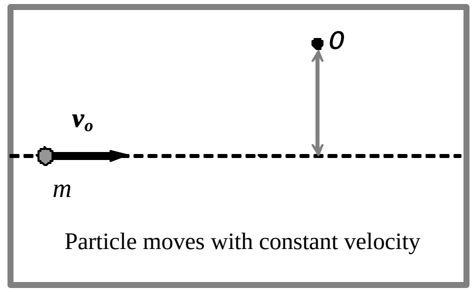
Homework: Angular momentum and Kepler's second law Name ______________________

1. The magnitude of any vector cross product can be written in terms of the magnitudes of the individual vectors and the sine of the angle f between those vectors: $\left|\vec{A}\times\vec{B}\right| = \left|\vec{A}\right|\left|\vec{B}\right|\sin f$.

   a. Explain in words <u>and</u> in diagrams why the magnitude of a vector cross product can be thought of as (i) the magnitude of $\vec{A}$ multiplied by the component of $\vec{B}$ that is perpendicular to $\vec{A}$, as well as (ii) the magnitude of $\vec{B}$ multiplied by the component of $\vec{A}$ that is perpendicular to $\vec{B}$.

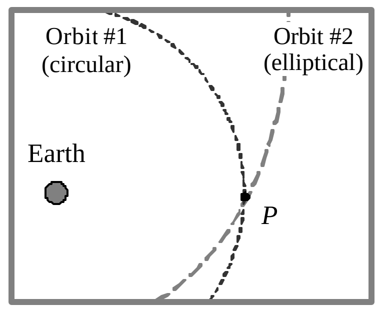
   b. Consider the situation shown at right, in which a particle of mass *m* moves with constant velocity.

   Apply your reasoning from part a to explain why the angular momentum of the particle (measured with respect to point *O*) <u>remains constant</u> while the particle moves along the straight line.



   c. Now consider two satellites, each of mass *m*, orbiting the Earth. One satellite (#1) follows a circular orbit, and the other (#2) follows an elliptical orbit that intersects the orbit of satellite #1 at point *P*, as shown at right. (Ignore the interactions between the satellites themselves.)

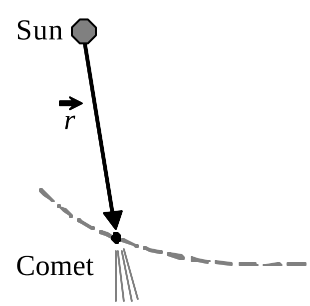
   If both satellites have the <u>same angular momentum</u>, which one passes through point *P* with the *greater speed?* Explain how you can apply your reasoning from part a to justify your answer.



2. A comet moves in a counter-clockwise orbit around the Sun. A portion of the orbit is shown below. (Ignore all gravitational forces acting on the comet other than that by the Sun.)

   a. The position $\vec{r}$ of the comet at a time *t* is shown in the diagram at right. In the diagram, draw a vector $d\vec{r}$ representing the infinitesimal displacement of the comet between time *t* and time *(t + dt)*.

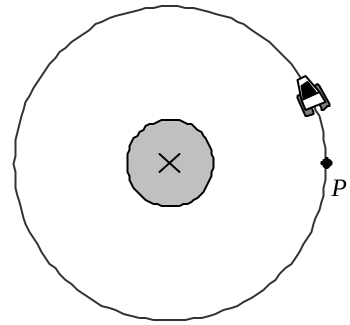
   b. Show that the magnitude of the vector cross-product $\vec{r}\times d\vec{r}$ is directly proportional to the area *dA* swept out by the comet from time *t* to time *(t + dt)*. Explain your reasoning in words <u>and</u> with one or more diagrams.



   (Hint: Show that the magnitude of $\vec{r}\times d\vec{r}$ is equal to the area of the parallelogram formed by $\vec{r}$ and $d\vec{r}$. How does the area *dA* swept out by the comet compare to that of the parallelogram?)

   c. On the basis of your work in part b, show that the rate *dA*/*dt* at which the comet sweeps out area along its orbit is directly proportional the angular momentum of the comet, and determine the constant of proportionality between the two quantities.

3. Consider a shuttle craft that follows a circular orbit around a planet (see diagram at right).



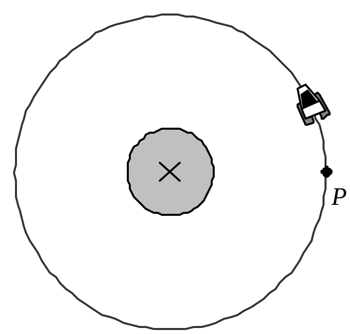
   a. Suppose that, the next time the shuttle passes point *P*, the pilot briefly fires the forward thrusters for a very short time, essentially causing the speed of the shuttle to suddenly <u>increase</u> at point *P*.

      i. Determine whether the sudden increase in the speed of the shuttle causes each of the following quantities to *increase*, *decrease*, or *remain unchanged*. Explain your reasoning in each case.

         - the angular momentum of the shuttle
         - the gravitational force on the shuttle at point *P*
         - the (centripetal) acceleration of the shuttle at point *P*
         - the radius of curvature of the orbit as measured at point *P*

      ii. On the diagram, make a qualitatively correct sketch of the new orbit of the shuttlecraft, and clearly indicate whether the latus rectum of the orbit is *larger than*, *smaller than*, or *the same as* before. (*Hint:* Point *P* will lie along the major axis of the new elliptical orbit.)

   b. Suppose instead that, the next time the shuttle passes point *P*, the pilot briefly fires a retrothruster for a very short time, essentially causing the speed of the shuttle to suddenly <u>decrease</u> at point *P*.



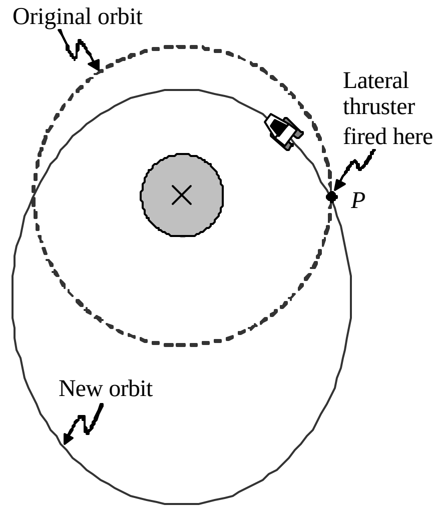
      i. Answer the same questions from part a(i) for the situation described here in part b. Clearly explain your reasoning.

      ii. On the diagram at right, make a qualitatively correct sketch of the new orbit of the shuttlecraft, and clearly indicate whether the latus rectum of the orbit is *larger than*, *smaller than*, or *the same as* before.

(Problem 3 is continued on the next page.)

3. [continued]

c. Suppose instead that, upon passing point *P*, the shuttle pilot were to fire a lateral thruster for a very short time. (Assume that the effect of firing the lateral thruster would be to suddenly add a component of velocity perpendicular to the original velocity.)



As a result, the shuttle is observed to follow a new elliptical orbit, shown at right. The semi-major axis of the new orbit is perpendicular to the imaginary line from point *P* to the center of the planet.

i. Did the sudden firing of the lateral thruster cause the speed of the shuttle to *suddenly increase, suddenly decrease,* or *remain unchanged?* Explain.

ii. Did the sudden firing of the lateral thruster cause the magnitude of the angular momentum of the shuttle to *suddenly increase, suddenly decrease, or remain unchanged?* Explain.

d. Reflect on your results in parts a – c of this problem. Do your results suggest a relationship between the angular momentum of the shuttle and the latus rectum of the shuttle's orbit? Use your results in parts a – c to defend your answer.