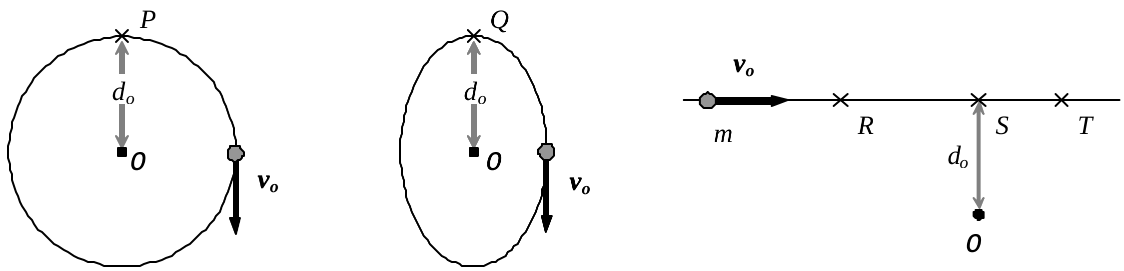
ANGULAR MOMENTUM AND KEPLER'S SECOND LAW

I. Angular momentum

The *angular momentum* of a point particle with position $\vec{r}$ and momentum $\vec{p}$ is equal to the vector cross-product of the position and momentum vectors: $\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}$.

In each top view diagram shown below, an object of mass *m* moves with *constant speed* v_o along the path shown. In each case the origin ***O*** of a coordinate system is located at a distance d_o from at least one point along the path.

Case 1 **Case 2** **Case 3**



A. With respect to the origin ***O*** shown in each case, compare and contrast the angular momentum of the object at points *P*, *Q*, and *S*. Discuss both magnitude and direction.

   How, if at all, would the angular momentum of the object in **case 3** be different if the origin ***O*** were to coincide exactly with point *S?* Explain.

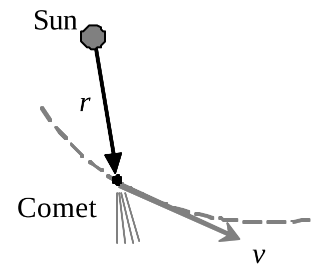
B. For **case 1** and **case 2,** state whether or not the angular momentum of the object (measured with respect to the origin ***O***) remains constant throughout the motion. (Recall that the particle moves with *constant speed* in all cases.)

C. Consider the motion of the object in **case 3.** Rank points *R, S,* and *T* according to the magnitude of the angular momentum of the object at those locations. Discuss your reasoning with your partners.

   (Hint: Consider the quantity $|\vec{r}|\sin q$ at the labeled points, where q is the angle between $\vec{r}$ and $\vec{p}$ *.)*

✓ **STOP HERE** and check your results with an instructor before proceeding to the next section.

II. Changes in angular momentum



A comet moves past the sun as shown. For simplicity, assume that the gravitational force on the comet by the sun is the only force on the comet.

We can express $d\vec{L}/dt$, the rate of change of the angular momentum of the comet, as:

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times m\vec{v}) = \left(\frac{d\vec{r}}{dt} \times m\vec{v}\right) + \left(\vec{r} \times m\frac{d\vec{v}}{dt}\right)$$

A. Justify the final expression written above for $d\vec{L}/dt$.

B. What can be said about the value of the *first term,* $\left(\frac{d\vec{r}}{dt} \times m\vec{v}\right)$, in the above expression? Explain.

C. Give an interpretation in your own words for the *second term,* $\left(\vec{r} \times m\frac{d\vec{v}}{dt}\right)$, in the above expression.

For the case of the comet, what can be said about the value of this term? Explain.

D. Summarize your results here in section II by answering the following questions:

1. In your own words, under what conditions will the angular momentum of an object change as time goes on? Be as concise and as specific as you can.

2. If a comet (or any other body) moves under the influence of only gravitational forces, what can be said about the rate of change of the angular momentum of that body? Explain.

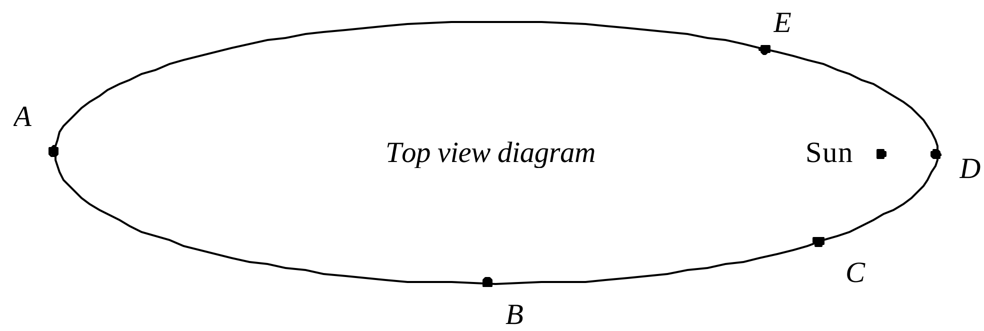
✓ **STOP HERE** and check your results with an instructor before proceeding to the next page.

Your work in section II proves an important law in celestial mechanics. This law, named after the German astronomer Johannes Kepler, is known as *Kepler's second law*. (*Note:* This law is also phrased in terms of the rate at which an orbiting body sweeps out area along its orbit.)

III. Application: Elliptical orbit of a comet

A comet orbits the sun counterclockwise along the elliptical orbit shown at right.

Several points *(A – E)* are labeled along the orbit. Points *A* and *D* are the points of farthest and closest approach, respectively.



A. At each labeled point *(A – E)*, draw and label arrows to indicate the directions of (i) the *velocity* and (ii) the *acceleration* of the comet at that point. Explain how you determined your answers.

B. Through which of the labeled points does the comet move with (i) the fastest speed? (ii) the slowest speed? Justify your answers **two** different ways:

- Use your results in part A and your knowledge of motion in two dimensions to determine how (if at all) the speed of the comet changes upon passing through each labeled point.

- Apply Kepler's second law in this situation.

C. Suppose that distance between point *A* and the sun is twice as long as the distance between point *B* and the sun.

1. Use Kepler's second law to explain why the speed of the comet at point *B* cannot be twice as great as its speed at point *A*.

2. Is the speed of the comet at point *B* *greater than* or *less than* twice as great as the speed at point *A?*