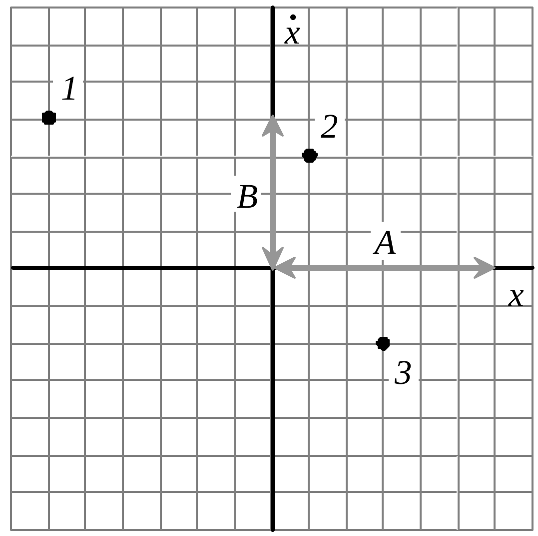
Consider an oscillator in which the damping force is non-linear, *i.e.*, the damping force is not simply proportional to the speed of the oscillating mass. The equation of motion for one such an oscillator is as follows (all constants are positive):

$$\ddot{x} + g'\left(\frac{x^2}{A^2} + \frac{\dot{x}^2}{B^2} - 1\right)\dot{x} + w_o^2 x = 0$$

The oscillator is set into motion from a variety of starting points *(1 – 3)* labeled in the phase space diagram at right. Note the parameters *A* and *B* labeled on the diagram.



Consider the motion of the oscillator immediately after it begins to move. **For each labeled starting point,** just after the oscillator begins to move:

i. Would the damping force experienced by the oscillator be exerted in *the same direction as its velocity, in the opposite direction from its velocity,* or *neither?* Explain how you can tell.

ii. Would the total energy of the oscillator *increase, decrease,* or *remain the same?* Explain how you can tell.

Point *1*	Point *2*	Point *3*
i.	i.	i.
ii.	ii.	ii.

PHASE SPACE DIAGRAMS: SELF-LIMITING OSCILLATORS

I. Equation of motion

Consider an oscillator in which the damping force is non-linear, *i.e.*, the damping force is not simply proportional to the speed of the oscillating mass. Oscillators of this type are called *non-linear* oscillators. The equation of motion for one example of such an oscillator is shown below (all constants are positive):

$$\ddot{x} + \gamma'\left(\frac{x^2}{A^2} + \frac{\dot{x}^2}{\beta^2 A^2} - 1\right)\dot{x} + \omega_o^2 x = 0 \qquad \text{(Eq. 1)}$$

We will compare the oscillator described above to a linearly damped oscillator having the same natural frequency ω_o of motion:

$$\ddot{x} + 2\gamma\dot{x} + \omega_o^2 x = 0 \qquad \text{(Eq. 2)}$$

A. In the linearly damped oscillator, is the damping force in the *same direction* or *opposite in direction* from the velocity? Explain how you can tell from the equation of motion.

B. Suppose that the initial conditions for the nonlinear oscillator were chosen such that the quantity $(x_o^2/A^2 + \dot{x}_o^2/\beta^2A^2)$ is greater than 1. Would the damping force be in the *same direction* or *opposite in direction* from the velocity? Explain how you can tell from the equation of motion.

How, if at all, would your answer change if the initial conditions were different such that $(x_o^2/A^2 + \dot{x}_o^2/\beta^2A^2)$ is less than 1? Explain your reasoning.

How, if at all, would your answer change if the initial conditions were different again such that $(x_o^2/A^2 + \dot{x}_o^2/\beta^2A^2)$ is equal to 1? Explain your reasoning.

C. For each of the cases described in part C above, would the total energy of the oscillator *increase*, *decrease*, or *remain constant* after it is released? Explain your reasoning for each case.

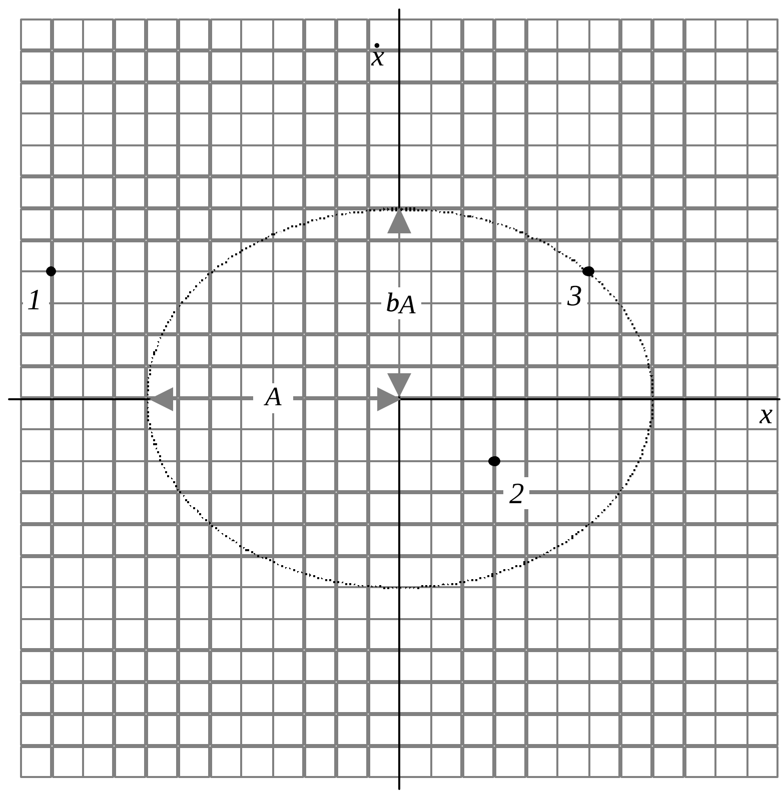
✓ **STOP HERE** and check your results with an instructor before proceeding to the next section.

II. Phase space trajectories of a self-limiting oscillator

Below we consider the phase space trajectory for a weakly damped oscillator *(i.e., g´* « w_o*)* for which the equation of motion is given in Eq. 1.

The phase space diagram below shows an ellipse with axes of length 2*A* (along the position axis) and 2*bA* (along the velocity axis). Three different initial starting points *(1, 2,* and *3)* are labeled.

A. For each labeled point, would the damping force exerted on the oscillator be in the *same direction as velocity, opposite in direction from velocity,* or *zero?* Explain.



B. Consider the case in which the oscillator begins from point *1*.

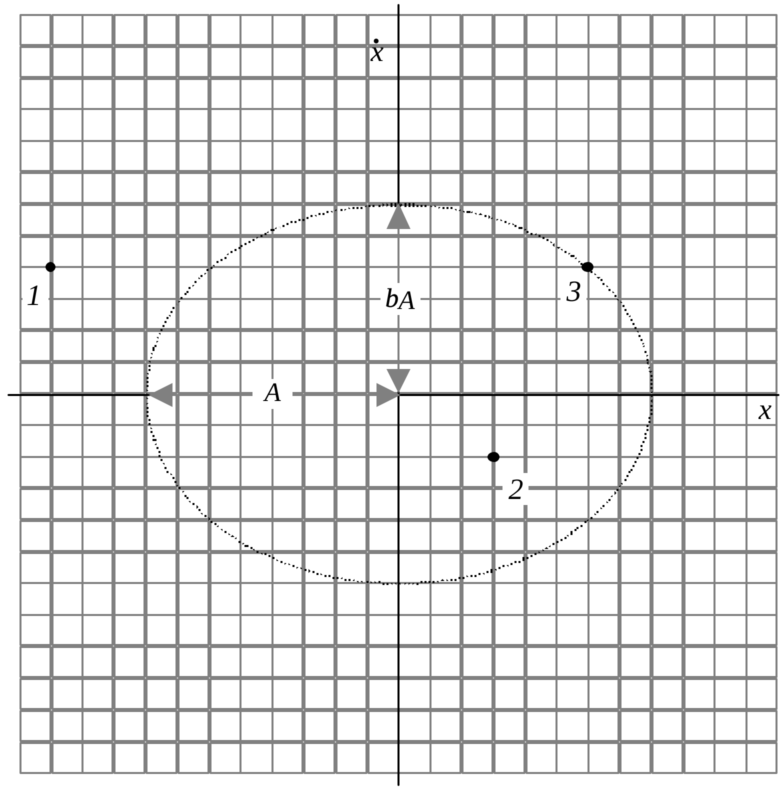
1. Sketch a qualitatively correct trajectory from point *1* that corresponds to approximately ¼ to ½ of an oscillation.

2. As the oscillator evolves, does the damping force exerted on the oscillator *increase* or *decrease* in magnitude? Explain how you can tell.

   On the basis of your answer, continue your phase space plot so that it shows at least two cycles of motion. Describe the asymptotic behavior of the oscillator (as $t \to \infty$) in words.

C. Repeat part B for the cases in which the oscillator instead starts to move from (i) point *2*, (ii) point *3*.

(The phase space diagram from the preceding page has been reproduced at right for convenience.)



D. The non-linear oscillator described by Eq. 1 is an example of a category of oscillators called *self-limiting oscillators*. The elliptical phase space trajectory shown above (having axes of length $2A$ and $2bA$) is called the *limit cycle* of the oscillator. In light of your results:

1. Explain why the terms *self-limiting oscillator* and *limit cycle* are appropriate.

2. Describe in words the significance of the parameter *b*. In particular, how is its meaning different from that of the natural frequency w_o?

Homework: ***Phase space diagrams: Self-limiting oscillators*** Name ______________________

Shown below are three different phase space diagrams. For each case below, identify which diagram could be used to represent that situation. If more than one diagram could apply in a particular case, **specify them all.** If none of the diagrams apply, state so explicitly. Explain your reasoning in each case.

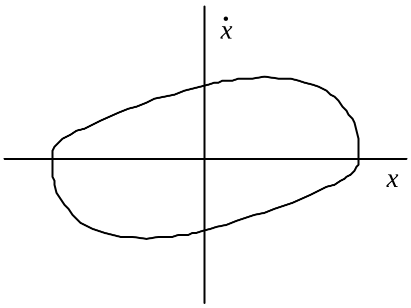


Diagram 1

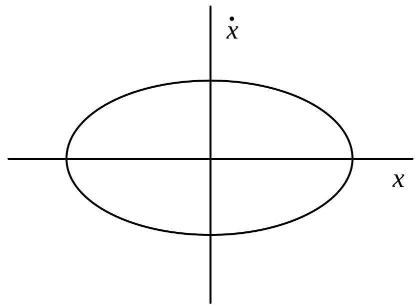


Diagram 2

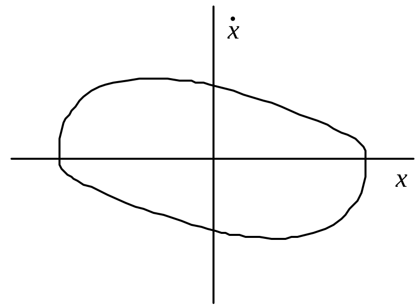


Diagram 3

a. the limit cycle of a non-linear oscillator of the form: $\ddot{x} + g\left(\frac{x^2}{A^2} + \frac{\dot{x}^2}{B^2} - 1\right)\dot{x} + w_o^2 x = 0 \quad (g > 0)$

b. the limit cycle of a non-linear oscillator of the form: $\ddot{x} + g\left(\frac{x^2}{A^2} - 1\right)\dot{x} + w_o^2 x = 0 \quad (g > 0)$

c. the limit cycle of a non-linear oscillator of the form: $\ddot{x} + g\left(\frac{\dot{x}^2}{B^2} - 1\right)\dot{x} + w_o^2 x = 0 \quad (g > 0)$